J. DIFFERENTIAL GEOMETRY
54 (2000) 227-253

NON-UNIVALENT HARMONIC MAPS HOMOTOPIC TO DIFFEOMORPHISMS

F.T. FARRELL, P. ONTANEDA & M.S. RAGHUNATHAN

We solve in this paper Problem 111 of the list compiled by S.-T. Yau in [32]. Here is a restatement of this problem.

Problem 111 of [32]. Let $f : M_1 \to M_2$ be a diffeomorphism between two compact manifolds with negative curvature. If $h : M_1 \to M_2$ is a harmonic map which is homotopic to f, is h a univalent map?

(This problem has recently been reposed in [31] as Grand Challenge Problem 3.6.) The answer to the problem was proven to be yes when $\dim M_1 = 2$ by Schoen-Yau [29] and Sampson [27]. Part of the interest in the problem comes from the fact that harmonic maps have become extremely useful in proving rigidity results; see for example [30], [6], [14], [33], [15] and [20]. Hence the negative answer given in this paper to Problem 111 places some limits on the applicability of the harmonic map techniques to rigidity questions. Our precise result is that for every integer $n \geq 6$ there is a pair of closed negatively curved Riemannian manifolds M_1 and M_2 with $\dim M_1 = n$, a diffeomorphism $f : M_1 \to M_2$, and a harmonic map $h : M_1 \to M_2$ homotopic to f such that h is not univalent (i.e., not a one-to-one map). Furthermore given any $\varepsilon > 0$, M_1 and M_2 can be constructed so that the sectional curvatures of M_2 are all pinched within ε of -1 and M_1 has constant -1 sectional curvatures.

This paper has evolved from the earlier papers [9], [21], [10], [11] and [12]. In fact, the crucial use made here of the Scharlemann-Siebenmann C^∞-Hauptvermutung [28] was earlier used in [12]. The second key ingredient is the existence of closed (real) hyperbolic manifolds with interesting cup product properties. Such manifolds are constructed in section

Received January 24, 2000. This research was supported in part by the National Science Foundation and CNPq, Brazil.

2 of this paper by elaborating on ideas in a letter written more than 10 years ago by one of the authors, M.S. Raghunathan, to W. Casselman. The construction is an extension of that contained in the joint work of J.J. Millson and M.S. Raghunathan [19] (The results in [19] were also crucially used in [21] and [11].)

1. Statement and proof of results

This section contains 4 results: Lemma, Corollary, Theorem and Addendum. We first state these results and then devote the rest of the section to proving them. Theorem and Addendum are the main results of the paper and were discussed in the introduction. Lemma and Corollary are used to prove Theorem and Addendum. Lemma is a consequence of the constructions done in Section 2 (in particular of 2.26) and Lemma is used to prove Corollary.

Lemma. *Given $m \in \mathbb{Z}$ (with $m \geq 6$) and $r \in \mathbb{R}^+$, there exists a pair of closed connected orientable (real) hyperbolic manifolds M and N and a pair of cohomology classes $\alpha \in H^1(M, \mathbb{Z}_2)$ and $\beta \in H^2(M, \mathbb{Z}_2)$ satisfying the following properties:*

1. *$dim M = m$ and N is a totally geodesic codimension-one submanifold of M whose normal geodesic tubular neighborhood has width $\geq r$.*
2. *The isometry class of N depends only on m (not on r).*
3. $\alpha \cup \beta \neq 0$.
4. *α is the Poincaré dual of the homology class represented by N* in *$H_{m-1}(M, \mathbb{Z}_2)$,*
5. *β is co-spherical; i.e; it is in the image of $H^2(S^2, \mathbb{Z}_2)$ under some continuous map $M \to S^2$.*

Corollary. *Given an integer $m \geq 6$ and a positive real number ε, there exists a m-dimensional closed connected orientable (real) hyperbolic manifold M and a homeomorphism $g : \mathcal{M} \to M$ with the following properties:*

1. *$\mathcal{M}$ is a negatively curved Riemannian manifold whose sectional curvatures are all in the interval $(-1-\varepsilon, -1+\varepsilon)$.*

2. *M and $\mathcal{M}$ are not PL homeomorphic.*

3. *There is a connected 2-sheeted covering space $\tilde{M} \to M$ such that $\tilde{g} : \tilde{\mathcal{M}} \to \tilde{M}$ is homotopic to a diffeomorphism.*

Remark. In property 3, $\tilde{\mathcal{M}} \to \mathcal{M}$ denotes the pullback of the covering space $\tilde{M} \to M$ via g, and $\tilde{g}$ is the induced homeomorphism making the diagram

$$\begin{array}{ccc} \tilde{\mathcal{M}} & \xrightarrow{\tilde{g}} & \tilde{M} \\ \downarrow & & \downarrow \\ \mathcal{M} & \xrightarrow{g} & M \end{array}$$

into a Cartesian square. Also, $\tilde{M}$ and $\tilde{\mathcal{M}}$ are given the differential structure and Riemannian metric induced by $\tilde{M} \to M$ and $\tilde{\mathcal{M}} \to \mathcal{M}$, respectively.

Theorem. *For every integer $m \geq 6$, there is a diffeomorphism $f : M_1 \to M_2$ between a pair of closed negatively curved m-dimensional Riemannian manifolds such that the (unique) harmonic map $h : M_1 \to M_2$ homotopic to f is not univalent.*

Addendum. *In the Main Theorem, either M_1 or M_2 can be chosen to be a real hyperbolic manifold and the other chosen to have its sectional curvatures pinched within ε of -1; where ε is any preassigned positive number.*

Proof of Theorem and Addendum. Let $g : \mathcal{M} \to M$ be the homeomorphism given by Corollary relative to m and ε. Set $M_1 = \tilde{\mathcal{M}}, M_2 = \tilde{M}$ and let $f : M_1 \to M_2$ be a diffeomorphism homotopic to $\tilde{g} : \tilde{\mathcal{M}} \to \tilde{M}$ which exists by property 3 of Corollary. Let $k : \mathcal{M} \to M$ be the unique harmonic map homotopic to g given by the fundamental existence result of Eells and Sampson [8] and uniqueness by Hartmann [13] and Al'ber [1]. Lifting this homotopy to the covering spaces $\tilde{\mathcal{M}}, \tilde{M}$ gives a smooth map

$$\tilde{k} : \tilde{\mathcal{M}} \to \tilde{M}$$

covering k and homotopic to $\tilde{g}$. Note that $\tilde{k}$ is also a harmonic map as is easily deduced from [7, 2.20 and 2.32]. Consequently, $\tilde{k}$ is the harmonic map $h : M_1 \to M_2$ mentioned in the statement of the Theorem. Also note that if $\tilde{k}$ is univalent, then so is k. Hence it suffices to show that k is *not* univalent. Since k is smooth, k univalent would mean that

$$k : \mathcal{M} \to M$$

is a C^∞-homeomorphism and hence M and $\mathcal{M}$ are PL-homeomorphic by the C^∞-Hauptvermutung proven by Scharlemann and Siebenmann [28]. And this would contradict property 2 of Corollary; consequently, k and hence also h are *not* univalent. This proves the Theorem and the part of the Addendum where M_2 is real hyperbolic.

To prove the case where M_1 is real hyperbolic; set $M_1 = \tilde{M}, M_2 = \tilde{\mathcal{M}}$ and let f be a diffeomorphism homotopic to $\tilde{g}^{-1}$. The rest of the argument is as before.

Proof of Corollary. Let M and N be as in Lemma relative to the given integer $m \geq 6$ and a sufficiently large positive real number r depending on ε. (How large is sufficient will presently become clear.) Note that the normal bundle of N in M is trivial since M and N are both orientable. The Riemannian manifold $\mathcal{M}$ is constructed by cutting M apart along N and reglueing with a twist determined by a certain self-diffeomorphism $f : N \to N$ and using Lemma 2.2 of [21]. (See [21, p. 10] for details of this construction.) Note that although M varies with the real number r, the Riemannian manifold N does not because of property 2 of Lemma. The number of possible self diffeomorphisms used for glueing (described below) will be finite; in fact this number is equal to the cardinality of $[N \times I, N \times \partial I; Top/O]$. (Here $I = [0,1]$ and $\partial I = \{0,1\}$.) Hence property 1 of Lemma shows that $\mathcal{M}$ will satisfy property 1 of Corollary provided r is chosen sufficiently large.

It remains to specify the finite set of glueing diffeomorphisms so that properties 2 and 3 are satisfied relative to a homeomorphism $g : \mathcal{M} \to M$. To do this we use smoothing theory as developed by Kirby-Siebenmann [16]. We start by associating to each element

$$\gamma \in [N \times I, N \times \partial I : Top/O]$$

a self-diffeomorphism

$$f_\gamma : N \to N$$

such that f_γ is topologically psuedo-isotopic to id_N; i.e., there exists a self-homeomorphism

$$F_\gamma : N \times [0,1] \to N \times [0,1]$$

such that, for all $x \in N$,

1. $F_\gamma(x, 0) = x$
2. $F_\gamma(x, 1) = f_\gamma(x)$.

This is done as follows. By smoothing theory, γ determines a pair (W, g) where W is a smooth manifold and $g : W \to N \times I$ is a homeomorphism which is a diffeomorphism over $N \times \partial I$. In particular W is a smooth s-cobordism. Now f_γ and F_γ are constructed using the s-cobordism theorem; see [21, §1] for more details. Then the set G of possible glueing maps is defined by

$$G = \{f_\gamma | \gamma \in [N \times I, N \times \partial I; Top/O]\}.$$

Let $\mathcal{M}_\gamma$ be M modified by the twist glueing determined by f_γ, and let

$$g_\gamma : \mathcal{M}_\gamma \to M$$

be the homeomorphism determined by F_γ. As pointed out above, since G is finite, there exists a real number $r_\varepsilon > 0$ such that each $\mathcal{M}_\gamma, \gamma \in G$, satisfies property 1 of Corollary provided M comes from choosing r in Lemma to be r_ε.

We next show how to use α and β to determine γ. For this we introduce some notation. Let

$$\omega : Top/O \to Top/PL$$

denote the canonical map, and let

$$\eta : S^3 \to Top/PL$$

and

$$\bar{\eta} : S^3 \to Top/O$$

denote the generators of $\pi_3(Top/PL)$ and $\pi_3(Top/O)$, respectively. Recall that both $\pi_3(Top/PL)$ and $\pi_3(Top/O)$ are cyclic groups of order 2 and that

$$\omega_\# : \pi_3(Top/O) \to \pi_3(Top/PL)$$

is an isomorphism. Hence both η and $\bar{\eta}$ are well defined up to homotopy and furthermore

$$\omega \circ \bar{\eta} \sim \eta.$$

Next, fix a continuous map

$$\hat{\beta} : M \to S^2$$

such that $(\hat{\beta})^*$ maps the generator of $H^2(S^2, \mathbb{Z}_2)$ to β. (This is possible because of Lemma's property 5.) Then identify S^1 with $I/\partial I$ and $N \times I$ with a tubular neighborhood of N in M. And define a continuous map

$$\hat{\alpha} : M \to S^1$$

by the formula

$$\hat{\alpha}(x) = \begin{cases} t & if \quad x = (y,t) \in N \times I, \\ \partial I & if \quad x \notin N \times I. \end{cases}$$

Let $\sigma, \triangle$ and ψ denote the inclusion map

$$\sigma : N \times I \to M,$$

the diagonal map

$$\triangle : M \to M \times M$$

and the canonical quotient map

$$\psi : S^2 \times S^1 \to S^2 \wedge S^1 = S^3.$$

Then the homotopy class $\gamma \in [N \times I, N \times \partial I; Top/O]$ determined by α and β is represented by the following composite map

$$N \times I \xrightarrow{\sigma} M \xrightarrow{\triangle} M \times M \xrightarrow{\hat{\beta}\times\hat{\alpha}} S^2 \times S^1 \xrightarrow{\psi} S^3 \xrightarrow{\bar{\eta}} Top/O.$$

Now define the Riemannian manifold $\mathcal{M}$ and the homeomorphism $g : \mathcal{M} \to M$ posited in Corollary by

$$\mathcal{M} = \mathcal{M}_\gamma \text{ and } g = g_\gamma.$$

It remains to verify Corollary's properties 2 and 3. To do this, let $\hat{\gamma} : M \to Top/O$ denote the composite map

$$M \xrightarrow{\triangle} M \times M \xrightarrow{\hat{\beta}\times\hat{\alpha}} S^2 \times S^1 \xrightarrow{\psi} S^3 \xrightarrow{\bar{\eta}} Top/O$$

and observe that the homotopy class of $\hat{\gamma}$, denoted $[\hat{\gamma}] \in [M, Top/O]$, corresponds to the smooth structure on M given by $g : \mathcal{M} \to M$; while $[\omega \circ \hat{\gamma}] \in [M, Top/PL]$ corresponds to the PL-structure given by the same map g relative to a Whitehead triangulation of $\mathcal{M}$.

Since Top/PL is a $K(\mathbb{Z}_2, 3)$ and $\eta \sim \omega \circ \bar{\eta}$, we see that this PL-structure on M is the homotopy class of the composite

$$M \xrightarrow{\triangle} M \times M \xrightarrow{\hat{\beta}\times\hat{\alpha}} S^2 \times S^1 \xrightarrow{\psi} S^3 \xrightarrow{\eta} K(\mathbb{Z}_2, 3).$$

Now a standard algebraic topology argument shows this composite considered as an element of $H^3(M, \mathbb{Z}_2)$ is $\alpha \cup \beta$. But $\alpha \cup \beta \neq 0$ by Lemma's property 2. Hence $g : \mathcal{M} \to M$ and $id_M : M \to M$ represent different

PL-structures on M since $0 \in H^3(M, \mathbb{Z}_2)$ corresponds to the standard PL-structure $id_M : M \to M$. Now the argument of [21, 3.1.2 and 3.1.3] shows that $\mathcal{M}$ and M are not PL-homeomorphic thus verifying property 2 of Corollary.

We next define the 2-sheeted cover $q : \tilde{M} \to M$ mentioned in Corollary to be the pullback via $(\hat{\beta} \times \hat{\alpha}) \circ \triangle$ of the 2-sheeted cover

$$id_{s^2} \times p : S^2 \times S^1 \to S^2 \times S^1,$$

where p is defined by

$$p(z) = z^2$$

for all $z \in S^1$. It is easily shown that $\tilde{M}$ is connected by using Lemma's property 4 and the fact that both M and N are connected. Now note, by naturality, that the smooth structure $\tilde{g} : \tilde{\mathcal{M}} \to \tilde{M}$ on $\tilde{M}$ corresponds to

$$[\hat{\gamma} \circ q] \in [\tilde{M}, Top/O].$$

Let $\varphi : S^3 \to S^3$ be a degree 2 map and $\xi : \tilde{M} \to S^2 \times S^1$ be the canonical map covering $(\hat{\beta} \times \hat{\alpha}) \circ \triangle$. Then we have the following homotopy commutative diagram:

$$\begin{array}{ccccccc}
\tilde{M} & \xrightarrow{\xi} & S^2 \times S^1 & \xrightarrow{\psi} & S^3 & & \\
q \downarrow & & \downarrow id_{S^2} \times p & & \downarrow \varphi & & \\
M & \xrightarrow{(\hat{\beta} \times \hat{\alpha}) \circ \triangle} & S^2 \times S^1 & \xrightarrow{\psi} & S^3 & \xrightarrow{\overline{\eta}} & Top/O
\end{array}$$

Consequently,

$$[\bar{\eta} \circ \varphi \circ \psi \circ \xi] = [\hat{\gamma} \circ q].$$

But $\bar{\eta} \circ \varphi$ is null homotopic; since $\pi_3(Top/O)$ has order 2 and φ is degree 2. Hence $\bar{\eta} \circ \varphi \circ \psi \circ \xi$ is also null homotopic. Therefore $\tilde{g} : \tilde{\mathcal{M}} \to \tilde{M}$ is topologically psuedo-isotopic to a diffeomorphism; thus verifying Corollary's property 3. This completes the proof of Corollary.

Proof of Lemma. The manifolds M and N result from contructions done in the next section; in particular from judiciously applying Corollary 2.26, whose set up starts in subsection 2.15.

Let $n = m, n_1 = m - 1$ and $n_2 = m - 2$ in this set up. Furthermore let $\mathbf{G}, \mathbf{L}, \mathbf{H}, \mathbf{G}_1, \mathbf{G}_2$ be the algebraic groups constructed in subsection 2.15 relative to this choice of integers n, n_1, n_2 and setting the algebraic number field $k = \mathbb{Q}(\sqrt{2})$. (Note $\mathbf{L}$ is constructed in 2.21.) Fix non-zero ideals $\underline{b}' \subset \underline{a}' \subset \mathbb{Z}$ as in lemma 2.22. Define a subgroup Λ of $\Gamma(\underline{a}')$ by

$$\Lambda = \Gamma(\underline{b}')(\Gamma(\underline{a}') \cap \mathbf{H}(\mathbb{Q}))$$

and set

$$\Lambda_1 = \Lambda \cap \mathbf{G}_1(\mathbb{Q})$$

(See subsection 2.1 for notation.) Fix also the non-zero ideal $\underline{b} \subset \underline{b}'$ posited in section 2.24 and let $\underline{c}$ denote any non-zero ideal of $\mathbb{Z}$ such that

$$\underline{c} \subset \underline{b}.$$

Set

$$\Phi = \Gamma(\underline{c})\Lambda_1 \tag{1}$$

and let

$$\Phi_i = \Phi \cap \mathbf{G}_i(\mathbb{Q}) \tag{2}$$

$i = 1, 2$. And note that

$$\Lambda_1 = \Phi_1. \tag{3}$$

We now define M and N as follows relative to a sufficiently small ideal $\underline{c}$ which depends on r:

$$\begin{aligned} M &= X/\Phi, \\ N &= X_1/\Phi_1. \end{aligned} \tag{4}$$

(How $\underline{c}$ is chosen will presently become clear.) Note that X and X_1 are isometric to $\mathbb{H}^m$ and $\mathbb{H}^{m-1}$, respectively, and that Φ consists of orientation preserving isometries of $\mathbb{H}^m$; cf. Remark 2.23 (iii) and Corollary 2.26. More precisely stated M and N are closed connected orientable (real) hyperbolic manifolds and that N is a totally geodesic codimension-one submanifold of M. The isometry class of N is clearly independent of $\underline{c}$, and hence of r, because of (3) and the second equation in (4). Also notice that the posited cohomology class α is determined by Lemma's property 4. To define β, set

$$T = X_2/\Phi_2. \tag{5}$$

Then T is a framable closed codimension-2 submanifold of M; hence it determines a co-spherical class $\beta \in H^2(M, \mathbb{Z}_2)$. Note that β is the Poincaré dual of the homology class represented by T in $H_{m-2}(M, \mathbb{Z}_2)$. Furthermore, the cup product $\alpha \cup \beta$ is the Poincaré dual of the homology class represented by the intersection $N^\cap T$ in $H_{m-3}(M, \mathbb{Z}_2)$. (Note that N and T intersect transversally.) And this homology class is different from zero because of Corollary 2.26. Hence Lemma's property 3 is verified.

It remains to pick the ideal $\underline{c}$ small enough so that the tubular neighborhood of N in M has width $\geq r$. As a first approximation, start by making the largest possible choice; i.e; set $\underline{c} = \underline{b}$ in (1) and call the resulting pair of closed hyperbolic manifolds thus obtained from (4) by M_0 and N. Let $\pi_1(M_0, N)$ denote the set of all free homotopy classes of maps of the closed interval $[0,1]$ into M_0 starting and ending in N. Each non-trivial such class is represented by a unique geodesic segment meeting N perpendicularly at its endpoints. Also this geodesic segment is a curve of minimal length in its free homotopy class. Furthermore, there are only finitely many such geodesic segments of length less than $2r$. Let $\gamma_1, \gamma_2, ..., \gamma_n$ list this set consisting of all (non-trivial) geodesic segments of length less than $2r$ which meet N perpendicularly at their endpoints. We may assume that $n \geq 1$. (Since otherwise we're done; because if $n = 0$, then the normal geodesic tubular neighborhood for N in M_0 has width $\geq r$.)

Note that $\pi_1(M_0, N)$ can be identified with the double coset space

$$\pi_1(N) \backslash \pi_1(M_0) / \pi_1(N) \ ;$$

therefore,

$$\pi_1(M_0, N) = \Phi_1 \backslash \Phi / \Phi_1. \tag{6}$$

Using (6) together with equations(1) and (3), there exist elements $g_1, g_2, ..., g_n$ in

$$\Gamma(\underline{b}) - \Lambda_1$$

such that the double coset containing g_i represents the free homotopy class of γ_i. A smaller non-zero ideal $\underline{c} \subset \underline{b}$ can be chosen such that

$$g_i \notin \Gamma(\underline{c})\Lambda_1 \tag{7}$$

for all $i = 1, 2, ..., n$. The ideal $\underline{c}$ is constructed using the fact that each g_i acts on $X = \mathbb{H}^m$ via elements

$$\bar{g}_i \in SO(f, v_0) - SO(f_1, v_0).$$

(See subsection 2.15 for this notation.) This is the non-zero ideal $\underline{c}$ we seek; i.e; set

$$M = X/(\Gamma(\underline{c})\Lambda_1).$$

Note that there can be no geodesic segments γ in M of length less than $2r$ which meets N perpendicularly at its endpoints; since such a γ would be a lift, relative to the covering projection $M \to M_0$, of one of the geodesic segments γ_i and thus contradict (7). Consequently the normal geodesic tubular neighborhood of N in M has width $\geq r$. This completes the proof of Lemma.

2. Construction of some hyperbolic manifolds

2.1. Let $\mathbf{G}$ be a connected, semisimple linear algebraic group over $\mathbb{Q}$, and $\mathbf{C}$ its centre. We fix once and for all an imbedding of $\mathbf{G}$ in some $GL(n)$ as a $\mathbb{Q}$-algebraic subgroup such that the following holds: for every prime p in $\mathbb{Z}, \mathbf{C}(\mathbb{Q}_p) \subset GL(n, \mathbb{Z}_p)$. If $\wedge$ is any $\mathbb{Q}$-algebra and $\mathbf{B}$ is any $\mathbb{Q}$-algebraic subgroup of $\mathbf{G}$, we denote by $\mathbf{B}(\wedge)$ the group of $\wedge$-points of $\mathbf{B}$ and identify it with a subgroup of $GL(n, \wedge)$ through the inclusion $\mathbf{B}(\wedge) \hookrightarrow \mathbf{G}(\wedge) \hookrightarrow GL(n, \wedge)$. We also set for any subring $\wedge' \subset \wedge, \wedge$ a $\mathbb{Q}$-algebra, $\mathbf{B}(\wedge') = \mathbf{B}(\wedge) \cap GL(n, \wedge')$, and set $\Gamma_{\mathbf{B}} = \mathbf{B}(\mathbb{Z}) (= \mathbf{B}(\mathbb{Q}) \cap GL(n, \mathbb{Z}))$. If $\underline{a} \subset \mathbb{Z}$ is an ideal, we set $\Gamma_{\mathbf{B}}(\underline{a}) = \{x \in \mathbf{B}(\mathbb{Z}) \mid x \equiv 1 (\text{mod } \underline{a}) - x$ is considered as an element of $GL(n, \mathbb{Z})\}$. If $\underline{a} = \mathbb{Z}$, then $\Gamma_{\mathbf{B}}(\mathbb{Z}) = \Gamma_{\mathbf{B}}$, and if $\underline{a} \neq \{0\}$, then $\Gamma_{\mathbf{B}}(\underline{a})$ has finite index in $\Gamma_{\mathbf{B}}$. We also set $\Gamma_{\mathbf{G}}(\underline{a}) = \Gamma(\underline{a})$ in the sequel. We denote by B the $\mathbb{R}$-points of $\mathbf{B}$. (All algebraic groups over subfields of $\mathbb{R}$ are denoted by bold-face capital Roman letters, and the corresponding standard letters will denote the $\mathbb{R}$-points). We fix a maximal compact subgroup $K \subset \mathbf{G}(\mathbb{R}) = G$ and denote the Riemannian symmetric space $K \backslash G$ by X. The group $\Gamma(\underline{a})$($\underline{a}$ a non-zero ideal in $\mathbb{Z}$) acts properly discontinuously on X. Let $\underline{a} = p_1^{r_1} \cdots p_\ell^{r_\ell}$ be the prime factorisation of $\underline{a}$ with $p_i, 1 \leq i \leq \ell$ distinct. For a prime p, let $M_p(r) = \{g \in GL(n, \mathbb{Z}_p) \mid g - 1 \equiv 0 \ (\text{mod } p^r)\}$. Then evidently if we set $\Omega(\underline{a}) = \prod_p M_p(r_p)$ where $r_p = r_i$ if $p = p_i$ and $r_p = 0$ otherwise, then $\Gamma(\underline{a}) = \Omega(a) \cap \Gamma$ for the diagonal inclusion $\Gamma \subset \prod_p M_p$. Let $\Omega^*(\underline{a}) = \Omega(\underline{a}) \prod_p \mathbf{C}(\mathbb{Q}_p)$ and $\Gamma^*(a) = \Gamma \cap \Omega^*(\underline{a})$.

2.2. Lemma. *Let p_0 be any prime. Then there is an integer $r \geq 0$ such that $\Gamma(p_0^\ell)$ is torsion free for all $\ell \geq r$. Any torsion element of $\Gamma^*(p_0^\ell)$ is in $\mathbf{C}(\mathbb{Q})$.*

Proof. If $\ell \geq r$ with r suitably large, the group $\{x \in GL(n, \mathbb{Z}_{p_0}) \mid x \equiv 1 \pmod{p_0^\ell}\}$ is torsion free; hence the first assertion. To see that the second assertion holds, observe first that $M_{p_0}(\ell)\mathbf{C}(\mathbb{Q}_{p_0})$ is the direct product of $M_{p_0}(\ell)$ and $\mathbf{C}(\mathbb{Q}_{p_0})$ for all $\ell \geq r$; this is because every element of $\mathbf{C}(\mathbb{Q}_p)$ is of finite order while $M_{p_0}(\ell)$ is torsion free. If $\gamma = \zeta.\alpha$ with $\zeta \in M_{p_0}(\ell)$, $\alpha \in \mathbf{C}(\mathbb{Q}_p)$ and ν is an integer such that $\alpha^\nu = 1$, then $\gamma^\nu = \zeta^\nu \in \Gamma \cap M_{p_0}(\ell) = \Gamma(p_0^\ell)$. It follows that if γ is a torsion element, then ζ is a torsion element and thus $\zeta = 1$. Hence $\gamma \in \mathbf{C}(\mathbb{Q}_p)$ and since $\gamma \in \mathbf{G}(\mathbb{Q}), \gamma \in \mathbf{C}(\mathbb{Q})$ proving our contension.

2.3. Suppose now that $\mathbf{B}$ is a connected reductive $\mathbb{Q}$-subgroup of $\mathbf{G}$ such that $B \cap K$ is a maximal compact subgroup of $B (= \mathbf{B}(\mathbb{R}))$. (In particular, we may take $\mathbf{B} = \mathbf{G}$). Then $Y = B \cap K \backslash B$ is in a natural fashion a totally geodesic Riemannian submanifold in X. One has evidently a natural map

$$Y/\Gamma_{\mathbf{B}}(\underline{a}) \to X/\Gamma(\underline{a})$$

for every non-zero ideal $\underline{a} \subset \mathbb{Z}$. The group $\mathbf{C}(\mathbb{R})$ acts trivially on X and if all torsion elements of $\Gamma^*(\underline{a})$ are contained in $\mathbf{C}(\mathbb{Q})$, then $\Gamma^*(\underline{a})/\mathbf{C}(\mathbb{Q}) = \overline{\Gamma}(\underline{a})$ acts fixed point freely on X. One has evidently a natural map $Y/\overline{\Gamma}_{\mathbf{B}}(\underline{a}) \to X/\overline{\Gamma}(\underline{a})$, where $\overline{\Gamma}_{\mathbf{B}}(\underline{a})$ is the image of $\Gamma^*_{\mathbf{B}}(\underline{a}) = \Gamma^*(\underline{a}) \cap \mathbf{B}(\mathbb{Q})$ in $\overline{\Gamma}(\underline{a})$, $\underline{a}$ being a non-zero ideal on $\mathbb{Z}$. The real Lie group B may not be connected and may contain elements which reverse the orientation on Y. Consequently for torsion free $\overline{\Gamma}_{\mathbf{B}}(\underline{a}), Y/\overline{\Gamma}_{\mathbf{B}}(\underline{a})$ is a manifold which may not be orientable in general. However, one has the following result due to Rohlfs and Schwermer [26]. (We have included a proof for the sake of completeness).

2.4. Lemma. *There exists a non-zero ideal $\underline{a} \subset \mathbb{Z}$ such that $\Gamma_{\mathbf{B}}(\underline{a}) \subset B^0 (=$ connected component of the identity in $B)$. More generally if $\mathbf{B}_i, 1 \leq i \leq \ell$ is any finite collection of reductive groups, we can find a non-zero ideal $\underline{a}$ such that $\Gamma_{\mathbf{B_i}}(\underline{a}) \subset B_i^0$ for all i. If the $\mathbf{B}_i$ are all semisimple, we can choose $\underline{a}$ to be coprime to any given non-zero ideal $\underline{b}$.*

Proof. Clearly the general case of finitely many $\mathbf{B}_i$ follows from the case of a single group $\mathbf{B}$. Consider first the case where $\mathbf{B}$ is semisimple. Let $p : \widetilde{\mathbf{B}} \to \mathbf{B}$ be the universal covering of $\mathbf{B}$; then $\widetilde{\mathbf{B}}$ is semisimple, $\widetilde{\mathbf{B}}$ has a natural definition over $\mathbb{Q}$, and p is a morphism over $\mathbb{Q}$. Let μ be the (finite) kernel of p. The exact sequence

$$1 \to \mu \to \widetilde{\mathbf{B}} \to \mathbf{B} \to 1$$

of $\mathbb{Q}$-groups gives rise to the following commutative diagram (for Galois cohomology) with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu(\mathbb{Q}) & \longrightarrow & \widetilde{\mathbf{B}}(\mathbb{Q}) & \longrightarrow & \mathbf{B}(\mathbb{Q}) & \xrightarrow{\delta_{\mathbb{Q}}} & H^1(\mathbb{Q},\mu) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mu(\mathbb{R}) & \longrightarrow & \widetilde{\mathbf{B}}(\mathbb{R}) & \longrightarrow & \mathbf{B}(\mathbb{R}) & \xrightarrow{\delta_{\mathbb{R}}} & H^1(\mathbb{R},\mu) \end{array}$$

We need only to find an ideal $\underline{a}$ in $\mathbb{Z}$ coprime to $\underline{b}$ such that $\delta_{\mathbb{R}}(\Gamma_{\mathbf{B}}(\underline{a})) = 0$, since $\widetilde{\mathbf{B}}$ being simply connected, $\widetilde{\mathbf{B}}(\mathbb{R})$ is connected. To do this observe first that $\Phi = \delta_{\mathbb{Q}}(\Gamma_{\mathbf{B}})$ is a finite group, since $H^1(\mathbb{Q},\mu)$ is an abelian torsion group while $\Gamma_{\mathbf{B}}$ is finitely generated. Now Φ being a finite set, we can find a (finite) Galois extension k of $\mathbb{Q}$ such that $\mu(\overline{\mathbb{Q}}) = \mu(k)$($\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$ containing k) and every element $\varphi \in \Phi$ can be represented by a 1-cocycle f_φ on $Gal(k/\mathbb{Q})$. This means that the image of f_φ in $H^1(k,\mu)$ is zero. It follows that every element of $\Gamma_{\mathbf{B}}$ then can be lifted to an element of $\widetilde{\mathbf{B}}(k)$. If k admits a real imbedding, every element of $\Gamma_{\mathbf{B}}$ would be in the image of $\widetilde{B}(\mathbb{R})$- and this last image is precisely B^0. Hence we have only to deal with the case where every archimedean completion of k is isomorphic to $\mathbb{C}$. If k_w is one such completion and σ_w is the complex conjugation in k_w, then the restriction of σ_w to k gives an element of $Gal(k/\mathbb{Q})$ which we continue to denote σ_w. Moreover, the σ_w as w varies over inequivalent archimedean valuations are all conjugates in $Gal(k/\mathbb{Q})$. Pick one such w and set $\sigma_w = \sigma$. Now by the Čebotarev density theorem ([18, Theorem 10, Ch. VIII]) there are infinitely many primes p all coprime to $\underline{b}$ such that for each of these primes, there is a completion k_v of k containing $\mathbb{Q}_p$ and unramified over it such that $(k_v : \mathbb{Q}_p) = 2$ and the unique nontrivial element in $Gal(k_v/\mathbb{Q}_p)$ restricts to σ on k. Fix one such prime p. Then the map $\widetilde{\mathbf{B}}(\mathbb{Q}_p) \to \mathbf{B}(\mathbb{Q}_p)$ maps $\widetilde{B}(\mathbb{Q}_p)$ onto an open subgroup. Thus we may assume that there is an integer $r > 0$ such that

$$\{x \in \mathbf{B}(\mathbb{Z}_p) \mid x \equiv 1 (mod\ p^r)\} \subset \ Image\ \widetilde{\mathbf{B}}(\mathbb{Q}_p).$$

In particular, $\Gamma_{\mathbf{B}}(p^r) \subset \ Image\ \widetilde{\mathbf{B}}(\mathbb{Q}_p)$ so that $\delta_{\mathbb{Q}_p}(\gamma)$ is zero (in $H^1(\mathbb{Q}_p,\mu)$). This means that if $\varphi \in \Phi$ is of the form $\delta_{\mathbb{Q}}(\gamma)$ with $\gamma \in \Gamma_{\mathbf{B}}(p^r)$, the image of $\delta_{\mathbb{Q}}(\gamma)$ in $H^1(\mathbb{Q}_p,\mu)$ is zero. But the Galois group of k_v over $\mathbb{Q}_p$ restricted to k is $<\sigma>$ and since $\mu(\overline{\mathbb{Q}}_p) = \mu(k)$ one concludes that $f_\varphi\ |_{<\sigma>}$ is cohomologous to zero where we have set $\varphi = \delta_{\mathbb{Q}}(\gamma), \gamma \in \Gamma_{\mathbf{B}}(p^r)$. It follows that $\delta_{\mathbb{Q}}(\gamma)$ has trivial image in $H^1(\mathbb{R},\mu)$ if $\gamma \in \Gamma_{\mathbf{B}}(p^r)$, i.e., $\gamma \in$ *Image* $\widetilde{\mathbf{B}}(\mathbb{R})$. This proves the lemma for semisimple $\mathbf{B}$.

For a general connected reductive $\mathbf{B}$, let $\mathbf{M} = [\mathbf{B},\mathbf{B}]$ and $\mathbf{T}$ the connected component of the identity in the centre of $\mathbf{B}$. Fix an ideal $\underline{a} \neq 0$ such that $\Gamma_{\mathbf{M}}(\underline{a}) \subset M^0$. From the fact that the congruence subgroup property holds for tori (this is essentially a theorem of Chevalley [5]; see also [25, Theorem 2.2]) one deduces that there is an ideal $\underline{a}' \neq 0, \underline{a}' \subset \underline{a}$ such that $\Gamma_{\mathbf{T}}(\underline{a}') \subset T^0$. ($T^0$ has finite index in T). Let $q : \mathbf{B} \to \mathbf{B}/\mathbf{M} = \mathbf{T}'$ be the natural morphism. Then $q(\Gamma_{\mathbf{T}}(\underline{a}'))$ is an arithmetic subgroup of $\mathbf{T}'$. It follows again from the theorem of Chevalley that if we fix a realisation of $\mathbf{T}'$ as a $\mathbb{Q}$-subgroup of some $GL(m)$, then $q(\Gamma_{\mathbf{T}}(\underline{a}')) \supset \Gamma_{\mathbf{T}'}(\underline{a}'')$ for a suitable non-zero ideal $\underline{a}'' \subset \underline{a}'$. Finally let $\underline{a}''' \subset \underline{a}''$ be a non-zero ideal such that $q(\Gamma_{\mathbf{B}}(\underline{a}''')) \subset \Gamma_{\mathbf{T}'}(\underline{a}'')$. We then claim that $\Gamma_{\mathbf{B}}(\underline{a}''') \subset B^0$. Let $\gamma \in \Gamma_{\mathbf{B}}(\underline{a}''')$. Then $q(\gamma) \in q(\Gamma_{\mathbf{T}}(\underline{a}'))$. Thus there is a $\delta \in \Gamma_T(\underline{a}')$ such that $\gamma\delta^{-1} \in \Gamma_{\mathbf{M}}(a')$, and $\gamma\delta^{-1} \in M^0$. On the other hand, $\delta \in T^0$ so that $\gamma \in M^0.T^0 = B^0$. Hence the lemma.

2.5. Since for any $\mathbf{B}$ as above, B^0 acts as orientation preserving diffeomorphisms on $Y = B \cap K\backslash B$, one sees that for a suitable non-zero ideal $\underline{a}'$ (coprime to a given ideal $\underline{b} \neq 0$ in $\mathbb{Z}$ if $\mathbf{B}$ is semisimple), the manifold $Y/\Gamma_{\mathbf{B}}(\underline{a})$ is orientable for any $\underline{a} \subset \underline{a}'$. Suppose now that $\mathbf{G}_1, \mathbf{G}_2$ are two connected $\mathbb{Q}$-subgroups such that $K_i = G_i \cap K$ is a maximal compact subgroup of G_i for $i = 1, 2$. Then $X_i = K_i\backslash G_i$ are connected totally geodesic (symmetric) sub-manifolds in X whose intersection Z is again a connected totally geodesic sub-manifold on which $G_1 \cap G_2 = H^1$ acts transitively. If $\mathbf{G}_1 \cap \mathbf{G}_2 = \mathbf{H}$, then H^1 has finite index in $H = \mathbf{H}(\mathbb{R})$ and contains the identity component H^0 of H. From the lemma, we conclude that, we can find an ideal $\underline{a}' \neq 0$ such that for any non-zero ideal $\underline{a} \subset \underline{a}'$, the manifolds $X/\Gamma(\underline{a}), X_i/\Gamma_{\mathbf{G_i}}(\underline{a})$ and $Z/\Gamma_{\mathbf{H}}(\underline{a})$ are all orientable. If $\mathbf{G}_1, \mathbf{G}_2$ and $\mathbf{H}$ are semisimple, $\underline{a}'$ can be chosen to be coprime to any preassigned ideal $\underline{b} \neq 0$. In the sequel we set $\Gamma_i(\underline{a}) = \Gamma_{\mathbf{G_i}}(\underline{a})$ and $\Gamma_{\mathbf{H}}(\underline{a}) = \Delta(a)$. The natural maps

$$Z/\Delta(\underline{a}) \to X_i/\Gamma_i(\underline{a}) \to X/\Gamma(\underline{a})$$

are immersions for $i = 1, 2$. In general they are however not imbeddings. We will presently show that the ideal $\underline{a}'$ above can be chosen in such a way that for all $\underline{a} \subset \underline{a}'$ these mappings are indeed imbeddings. Towards this end we prove

2.6. Lemma. *Let $\mathbf{B}_1, \mathbf{B}_2$ be two connected reductive algebraic $\mathbb{Q}$-subgroups of $\mathbf{G}$ and $B_i = \mathbf{B}_i(\mathbb{R})$ $i = 1, 2$. Then there is an ideal $\underline{a}' \subset \mathbb{Z}$*

coprime to any given non-zero ideal $\underline{b} \subset \mathbb{Z}$ *such that if* $B_1\gamma \cap KB_2 \neq \phi$ *for* $\gamma \in \Gamma(\underline{a})$ *with* $\underline{a} \subset \underline{a}'$, *then* $\gamma \in \mathbf{B}_1.\mathbf{B}_2$.

Proof. We observe first that if $k \in K$, the $(\mathbf{B}_1, \mathbf{B}_2)$ double coset $\mathbf{B}_1 k \mathbf{B}_2$ is a closed subvariety of $\mathbf{G}$. This follows from the results of Birkes [3]. Birkes shows that if $\mathbf{M}$ is a reductive *anisotropic* algebraic group over $\mathbb{R}$ and ρ is representation of $\mathbf{M}$ defined over $\mathbb{R}$, on a vector space V then the $\mathbf{M}$ orbit of any vector in $V(\mathbb{R})$ is closed. To apply this result, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G with $\mathfrak{k}$ being the subalgebra corresponding to K and $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$. Let $\mathfrak{b}_i, i = 1, 2$ be the subalgebras of $\mathfrak{g}$ corresponding to the $B_i, \mathfrak{k}_i = \mathfrak{k} \cap \mathfrak{b}_i$ and $\mathfrak{q}_i = \mathfrak{p} \cap \mathfrak{b}_i$ so that $\mathfrak{b}_i = \mathfrak{k}_i \oplus \mathfrak{q}_i$ is a Cartan - decomposition of $\mathfrak{b}_i$. Let $\mathfrak{g}' = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{p}, \mathfrak{b}_i' = \mathfrak{k}_i \oplus \sqrt{-1}\mathfrak{q}_i$; these are then compact Lie algebras over $\mathbb{R}$ and define corresponding anisotropic $\mathbb{R}$-forms $\mathbf{G}', \mathbf{B}_1', \mathbf{B}_2'$ of $\mathbf{G}, \mathbf{B}_1, \mathbf{B}_2$ respectively. Moreover there is a natural isomorphism Φ (over $\mathbb{C}$) of $\mathbf{G}$ on $\mathbf{G}'$ which carries $\mathbf{B}_i$ onto $\mathbf{B}_i'$ and induces identity on K (K has natural inclusions in G and $G' = \mathbf{G}'(\mathbb{R})$; the latter is induced by the inclusion of $\mathfrak{k}$ in $\mathfrak{g}' = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{p}$). We fix an identification of $\mathbf{G}'$ as a $\mathbb{R}$-subgroup of $GL(V)$ for some vector space V over $\mathbb{R}$. We then have a natural action of $\mathbf{B}_1' \times \mathbf{B}_2'$ on *End* V given by $T \mapsto b_1 T b_2$ where $T \in End\ V(\mathbf{C}), b_i \in \mathbf{B}_i'$. Since $\mathbf{B}_i'$ are anisotropic over $\mathbb{R}$, Birkes' result tells us that if $T \in End\ V(\mathbb{R}), \mathbf{B}_1' T \mathbf{B}_2'$ is a closed subvariety in $End\ V(\mathbb{C})$ and hence in $\mathbf{G}'$. It follows that if $T \in \mathbf{G}'(\mathbb{R}), \mathbf{B}_1' T \mathbf{B}_2'$ is closed in $\mathbf{G}'$. Thus if $T \in \mathbf{G}'(\mathbb{R}), \mathbf{B}_1 \Phi^{-1}(T) \mathbf{B}_2$ is closed in $\mathbf{G}$. Since Φ is the identity morphism on K, the double coset $\mathbf{B}_1 k \mathbf{B}_2$ is closed for *any* $k \in K$. Now let $\mathbb{Q}[\mathbf{G}]$ denote the coordinate ring of $\mathbf{G}$ over $\mathbb{Q}$ and I the subalgebra of $(\mathbf{B}_1 \times \mathbf{B}_2)$ invariants for the action $(g_1, g_2)g = g_1 g g_2^{-1}$ for $g \in \mathbf{G}, g_i \in \mathbf{B}$, in $\mathbb{Q}[\mathbf{G}]$. Then I is a finitely generated $\mathbb{Q}$-algebra. Let $S \subset I$ be a finite set of generators for I. We assume as we may that all $f \in I$ take the value 0 at $1 \in \mathbf{G}$. We have fixed a realisation of $\mathbf{G}$ as a $\mathbb{Q}$- subgroup of $GL(n)$ (for some n). Let $\lambda_{ij} \in \mathbb{Q}[\mathbf{G}]$ be the function that assigns to each $g \in \mathbf{G}$, the value $a_{ij}(g) - \delta_{ij}$ where $\{a_{ij}(g) \mid 1 \leq i, j \leq n\}$ is the matrix of g. Then the λ_{ij} generate $\mathbb{Q}[\mathbf{G}]$ so that every $f \in \mathbb{Q}[\mathbf{G}]$ may be expressed as a polynomial $P_f(\{\lambda_{ij} \mid 1 \leq i, j \leq n\})$ in the λ_{ij} with coefficients in $\mathbb{Q}$; $f(1) = 0$ if and only if P_f has constant term equal to zero: this holds in particular for $f \in S$. Now since K is compact, there is a positive integer N coprime to $\underline{b}$ such that one has $| f(K) | < N$ for all $f \in S$. We assume - as we may - by replacing the f by integral multiples if need be - that f is a polynomial in the $\lambda_{ij}, 1 \leq i, j \leq n$, with integral coefficients. It then follows that if $\gamma \in \Gamma(\underline{a}')$ (so that $\lambda_{ij}(\gamma) \in \underline{a}'), f(\gamma) \in \underline{a}'$ for all $f \in S$. If we take $\underline{a}'$ to be contained in N, we see that $f(\gamma)$ is an integer divisible by N. On the other hand if γ is such that $B_1\gamma \cap kB_2 \neq \phi$ with $k \in K$, then $\gamma \in \mathbf{B}_1 k \mathbf{B}_2$ so that $f(\gamma) = f(k)$ leading to $\mid f(\gamma) \mid < N$. This means that $f(\gamma) = 0$ for all $f \in S$; and since S generates I as a $\mathbb{Q}$-algebra, $f(\gamma) = 0$ for *all* $f \in I$ with $f(1) = 0$. Now the orbits $\mathbf{B}_1.\mathbf{B}_2$ and $\mathbf{B}_1 k \mathbf{B}_2 (k \in K)$ are both closed. Consequently if they are distinct, one can find a $f \in I$ with $f(1) = 0$ but $f(k) \neq 0$. Thus $\mathbf{B}_1.\mathbf{B}_2 = \mathbf{B}_1 k \mathbf{B}_2$. Hence if $\gamma \in \Gamma(\underline{a})$ with $\underline{a} \subset \underline{a}'$ is such that $B_1\gamma \cap KB_2 \neq \phi$, then $\gamma \in \mathbf{B}_1.\mathbf{B}_2$. This proves the lemma.

2.7. Corollary. *The notations are as in 2.5. Then given an ideal* $\underline{b} \neq 0$ *in* $\mathbb{Z}$, *there is an ideal* $\underline{a}'$ *coprime to* $\underline{b}$ *such that for any* $\gamma \in \Gamma(\underline{a})$ *with* $\underline{a} \subset a'$, *if* $X_i\gamma \cap X_i \neq \phi$ *then* $\gamma \in G_i$. *Also if* $Z\gamma \cap Z \neq \phi$ *for* $\gamma \in \Gamma(\underline{a}), \gamma \in H$.

Proof. We need only take $\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{G}_i$ in Lemma 2.6. to prove the first assertion. For the second take $\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{H}$.

2.8. From now on we assume that $\mathbf{G}$ is *anisotropic* over $\mathbb{Q}$. It follows that any reductive $\mathbb{Q}$-subgroup $\mathbf{B}$ of $\mathbf{G}$ is also anisotropic over $\mathbb{Q}$ and that $B/\Gamma_{\mathbf{B}}$ is compact [4]. As before we fix $\mathbb{Q}$-subgroups $\mathbf{G}_1, \mathbf{G}_2$ of $\mathbf{G}$ and a maximal compact subgroup K in $G = \mathbf{G}(\mathbb{R})$ such that $G_i \cap K = K_i$ is a maximal compact subgroup of $G_i = \mathbf{G}_i(\mathbb{R})$. We set $X_i = K\backslash G_i, i = 1, 2$ and identify $X_i, i = 1, 2$ as connected totally geodesic submanifolds of X. Let $Z = X_1 \cap X_2$ so that Z is also a connected totally geodesic submanifold in X. Let $\mathbf{H} = \mathbf{G}_1 \cap \mathbf{G}_2$; then $H = \mathbf{H}(\mathbb{R})$ acts transitively on Z. Since $G_i/\Gamma_{\mathbf{G}_i}$ and $H/\Gamma_{\mathbf{H}}$ are compact, the quotients $X/\Gamma, X_i/\Gamma_{\mathbf{G_i}}$ and $Z/\Gamma_{\mathbf{H}}$ are all compact. As before, we set $\Gamma_i = \Gamma_{\mathbf{G}_i}$ and $\Delta = \Gamma_{\mathbf{H}}$ and for an ideal $\underline{a} \neq 0$ in $\mathbb{Z}$, set $\Gamma_i(\underline{a}) = \Gamma_i \cap \Gamma(\underline{a})$ and $\Delta(\underline{a}) = \Delta \cap \Gamma(\underline{a})$. We also assume that $dimZ = dimX_1 + dimX_2 - dimX$-equivalently X_1 and X_2 intersect transversally. We fix a non-zero ideal $\underline{a}' \subset \mathbb{Z}$ such that the following conditions are satisfied:

let $\Phi \subset \Gamma(\underline{a}')$ be any subgroup of finite index, $\Phi_i = \Phi \cap \Gamma_i(\underline{a}'), i = 1, 2$ and $\Psi = \Phi \cap \Delta(\underline{a}')$; then

(i) Φ is torsion free;

(ii) if $\gamma \in \Phi$ (resp. $\Phi_i, i = 1, 2$, resp. Ψ), then γ acting on X (resp. $X_i, i = 1, 2$, resp. Z) is orientation preserving;

(iiii) the map $X_i/\Phi_i \to X/\Phi, i = 1, 2$ and $Z/\Psi \to X_i/\Phi_i$ are smooth imbeddings.

(iv) $\Phi \cap G_1 K G_2 (= \Phi \cap G_1^0 K G_2^0) \subset \mathbf{G}_1.\mathbf{G}_2$ (here G_i^0 = identity connected component of G_i).

We now wish to examine the intersection of the submanifolds $X_i/\Phi_i, i = 1, 2$ in X/Φ with $\Phi, \Phi_i, i = 1, 2$ as above. Let $p = p_\Phi : X \to X/\Phi$ be the natural projection. If $x_0 \in X_1/\Phi_1 \cap X_2/\Phi_2$, we can find $\widetilde{x}_1 \in X_1$ and $\gamma \in \Phi$ such that $\widetilde{x}_1\gamma = \widetilde{x}_2 \in X_2$ and $p(\widetilde{x}_1) = x_0 (= p(\widetilde{x}_2))$; but this means that we can find $g_i \in G_i$ for $i = 1, 2$ and $k \in K$ such that

$$g_1 k g_2 = \gamma.$$

Conversely, if $\gamma \in G_1 K G_2$, it is clear that $\overset{\bullet}{e}\, g_1^{-1}\gamma = \overset{\bullet}{e}\, g_2$ where $\overset{\bullet}{e} \in X$ is the identity coset. It is now immediate that

$$X_1/\Phi_1 \cap X_2/\Phi_2 = p(\cup_{\gamma \in G_1 K G_2 \cap \Phi}\ (X_1\gamma \cap X_2)).$$

Observe that our choice of $\underline{a}'$ has been made to ensure that

$$G_1 K G_2 \cap \Phi \subset \mathbf{G}_1 \mathbf{G}_2.$$

Thus if we want the intersection $X_1/\Phi_1 \cap X_2/\Phi_2$ to be connected, it suffices to demand that for any $\gamma \in \Phi$,

$$X_1\gamma \cap X_2 \subset (X_1 \cap X_2)\theta$$

with $\theta \in \Phi$ (if $\theta \in \Phi$, from the injectivity of X_2/Φ_2 in X/Φ, one sees that θ is in fact in Φ_2).This would mean in fact that $X_1/\Phi_1 \cap X_2/\Phi_2 = Z/\Psi$. We will show that it is possible to choose $\Phi \subset \Gamma(\underline{a}')$ so that this condition can in fact be met provided $\mathbf{G}_1, \mathbf{G}_2$ and $\mathbf{H}$ satisfy certain conditions.

2.9. We fix once and for all an ideal $\underline{a}' \neq 0$ in $\mathbb{Z}$ as in 2.8. Let $\wedge \subset \Gamma(\underline{a}')$ be a subgroup (of finite index) such that $\wedge \supset \Gamma(\underline{c})$ for some non-zero ideal $\underline{c}$ in $\mathbb{Z}$. Let $\wedge_i = \wedge_i = \wedge \cap \underline{G}_i(\mathbb{Q})$. Now according to a theorem due to Borel and Harish-Chandra [4], the set $D(\wedge)$ of double cosets $\wedge_1 \backslash \wedge \cap \mathbf{G}_1 \mathbf{G}_2 / \wedge_2$ is finite. It follows that there is an ideal $\underline{b}(\wedge) = \underline{b} \neq 0$ in $\mathbb{Z}$ with the following property: $\Gamma(\underline{b}) \subset \wedge$, and for any non-zero ideal $\underline{b}' \subset \underline{b}$, with $\Gamma(\underline{b}') \subset \wedge$ the image $D(\underline{b}')$ of $\Gamma(\underline{b}') \cap \mathbf{G}_1\mathbf{G}_2$ in $D(\wedge)$ equals $D(\underline{b})(=$ image of $\Gamma(\underline{b}) \cap \mathbf{G}_1\mathbf{G}_2)$. We fix an ideal $\underline{b} = \underline{b}(\wedge)$ with this property with $\wedge$ as above. Let $\mathbf{G}(\mathbb{A}_f)$ denote the adéle group of $\mathbf{G}$ formed out of all the non-archimedian valuations. Let Φ

be a subgroup of (finite index in) $\wedge$ such that $\Phi \subset \wedge_1\Gamma(\underline{b})$ and such that $\Phi \supset \wedge_1\Gamma(\underline{c})$ ($\wedge_1 = \wedge \cap \mathbf{G}_1(\mathbb{Q})$) for some nonzero ideal $\underline{c}$. Let $\Phi_i = \Phi \cap \mathbf{G}_i(\mathbb{Q}), i = 1, 2$, and let $\overset{\wedge}{\Phi}$ (resp. $\overset{\wedge}{\Phi}_i, i = 1, 2$) be the closure of Φ (resp. $\Phi_i, i = 1, 2$) in the adéle group $\mathbf{G}(\mathbb{A}_f)$. Note that $\overset{\wedge}{\Phi}$ (resp. $\overset{\wedge}{\Phi}_i, i = 1, 2$) has a natural identification with the projective limit of the groups $\{\Phi/\Gamma(\underline{c}) \mid \underline{c}$ a nonzero ideal such that $\Gamma(\underline{c}) \subset \Phi\}$ (resp. $\{\Phi_i/\Gamma_i(\underline{c}) \mid \underline{c}$ a nonzero ideal such that $\Gamma_i(\underline{c}) \subset \Phi_i\}, i = 1, 2$). With this notation we have the following:

2.10. Lemma. $\Phi \cap \mathbf{G}_1\mathbf{G}_2 \subset \overset{\wedge}{\Phi}_1 \overset{\wedge}{\Phi}_2$.

Proof. We have here identified $\mathbf{G}(\mathbb{Q})$ as a subgroup of $\mathbf{G}(\mathbb{A}_f)$. Suppose now that $\gamma \in \Phi \cap \mathbf{G}_1(\mathbb{C})\mathbf{G}_2(\mathbb{C})$. Let $\underline{c}_n, 1 \le n < \infty$ be a decreasing sequence of nonzero ideals which is cofinal in the family of all non-zero ideals in $\mathbb{Z}$. We assume that $\underline{c}_n \subset \underline{b}$ and that $\Gamma(\underline{c}_n) \subset \Phi$. Then from the choice of $\underline{b}$ it follows (since $\Phi \subset \wedge_1\Gamma(\underline{b})$) that for every $n, 1 \le n < \infty$, we can find $\gamma_i(n) \in \wedge_i, i = 1, 2$, and $\gamma(n)$ in $\Gamma(\underline{c}_n)$ such that $\gamma = \gamma_1(n)\gamma(n)\gamma_2(n)$. Now since $\wedge_1 \subset \Phi$, we see that $\gamma_1(n) \in \Phi_1$, and $\gamma_2(n) = \gamma(n)^{-1}\gamma_1(n)^{-1}\gamma \in \Phi \cap \wedge_2 = \Phi_2$. Since $\overset{\wedge}{\Phi}_1$ and $\overset{\wedge}{\Phi}_2$ are compact, we can find a sequence $\lambda(n), 1 \le n < \infty$ of integers such that $\gamma_i(\lambda(n))$ tends to a limit $\overset{\wedge}{\gamma}_i \in \Phi_i, i = 1, 2$. On the other hand, since $\underline{c}_n, 1 \le n < \infty$ is cofinal in the family of all non-zero ideals, $\gamma(n)$ tends to the identity element. Thus $\gamma = \overset{\wedge}{\gamma}_1 \overset{\wedge}{\gamma}_2$ with $\overset{\wedge}{\gamma}_i \in \overset{\wedge}{\Phi}_i, i = 1, 2$. Hence the lemma.

2.11. Let $\overline{\mathbb{Q}}$ denote an algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and $\mathcal{G}$ the Galois group of $\overline{\mathbb{Q}}$ over $\mathbb{Q}$. Let $\gamma \in \Gamma \cap \mathbf{G}_1(\mathbb{C})\mathbf{G}_2(\mathbb{C})$; then $\gamma \in \Gamma \cap \mathbf{G}_1(\overline{\mathbb{Q}})\mathbf{G}_2(\overline{\mathbb{Q}})$ (nullstellensatz), i.e., $\gamma = g_1g_2$ with $g_i \in G_i(\overline{\mathbb{Q}}), i = 1, 2$. One then has $\sigma(g_1)\sigma(g_2) = \sigma(\gamma) = \gamma = g_1g_2$ for all $\sigma \in \mathcal{G}$. Hence $A_\sigma(\gamma) = g_1^{-1}\sigma(g_1) = g_2\sigma(g_2)^{-1}$ is in $(\mathbf{G}_1 \cap \mathbf{G}_2)(\overline{\mathbb{Q}})$; and $\sigma \mapsto A_\sigma(\gamma)$ is a 1- cocycle on $\mathcal{G}$ with values in $(\mathbf{G}_1 \cap \mathbf{G}_2)(\overline{\mathbb{Q}})$. The element $c(\gamma)$ in $H^1(\mathbb{Q}, \mathbf{H})$ determined by the 1-cocycle is easily seen to be independent of the decomposition $\gamma = g_1g_2$. Moreover, the very definition of the cocycle $\{A_\sigma(\gamma), \sigma \in \mathcal{G}\}$ shows that the image of $c(\gamma)$ in $H^1(\mathbb{Q}, \mathbf{G}_i)$ is trivial for $i = 1, 2$. Suppose now that $\gamma \in \Phi$ with Φ as in 2.9. Then by Lemma 2.10 we see that $c(\gamma)$ has trivial image in $H^1(\mathbb{Q}_p, \mathbf{H})$ for every $p \in \mathcal{P}$. Therefore we have the following lemma:

2.12. Lemma. *Suppose* $\mathbf{G}_1, \mathbf{G}_2, \mathbf{H}$ *are such that the fibre over* *the trivial element in* $H^1(\mathbb{Q}, \mathbf{G}_1) \times H^1(\mathbb{Q}, \mathbf{G}_2) \times \prod_{p\in\mathcal{P}} H^1(\mathbb{Q}_p, \mathbf{H})$ *for the natural map*

$$H^1(\mathbb{Q}, \mathbf{H}) \to H^1(\mathbb{Q}, \mathbf{G}_1) \times H^1(\mathbb{Q}, \mathbf{G}_2) \times \prod_{p\in\mathcal{P}} H^1(\mathbb{Q}_p, \mathbf{H})$$

is trivial. Then for Φ *as in 2.9,* $\Phi \cap \mathbf{G}_1\mathbf{G}_2 \subset \mathbf{G}_1(\mathbb{Q}).\mathbf{G}_2(\mathbb{Q})$.

Proof. The assumptions guarantee that $c(\gamma)$ is trivial in $H^1(\mathbb{Q}, \mathbf{H})$. This means that we can find $h \in H(\overline{\mathbb{Q}})$ such that $g_1^{-1}\sigma(g_1) = A_\sigma(\gamma) = h^{-1}\sigma(h)$ for all $\sigma \in \mathcal{G}$ leading to $g_1h^{-1} = \sigma(g_1h^{-1})$ for all $\sigma \in \mathcal{G}$. Thus $u_1 = g_1h^{-1} \in \mathbf{G}_1(\mathbb{Q})$. Analogously, $u_2 = hg_2 \in \mathbf{G}_2(\mathbb{Q})$ so that $\gamma = u_1u_2 \in \mathbf{G}_1(\mathbb{Q}).\mathbf{G}_2(\mathbb{Q})$.

2.13. Once again let us fix a Φ as in 2.9. Then one has, assuming that the triple $(\mathbf{G}_1, \mathbf{G}_2, H)$ satisfies the conditions of Lemma 2.12, that any $\gamma \in \Phi \cap \mathbf{G}_1(\mathbb{C})\mathbf{G}_2(\mathbb{C})$ can be expressed as a product $\gamma = g_1g_2$ with $g_i \in \mathbf{G}_i(\mathbb{Q})$. On the other hand, we have $\gamma = \hat{\gamma}_1\hat{\gamma}_2$ with $\hat{\gamma}_i \in \hat{\Gamma}_i, i = 1, 2$. We conclude, therefore, that

$$(*) \qquad \hat{\gamma}_1^{-1} g_1 = \hat{\gamma}_2\, g_2^{-1} (\in \mathbf{H}(\mathbb{A}_f)).$$

2.14. Lemma. *Suppose now that* $\gamma, g_1, g_2, \hat{\gamma}_1$ *and* $\hat{\gamma}_2$ *are as above and that* $\hat{\gamma}_1^{-1} g_1$ *is in the closure of* $\mathbf{H}(\mathbb{Q})$ *in* $\mathbf{H}(\mathbb{A}_f)$. *Then* $\gamma = \gamma_1\gamma_2$ *with* $\gamma_i \in \Phi_i$.

Proof. Observe that there is an open (and closed) subgroup Ω of $\mathbf{G}(\mathbb{A}_f)$ such that $\Omega \cap \mathbf{G}(\mathbb{Q}) = \Phi$. (And hence $\Omega \cap \mathbf{G}_i(\mathbb{Q}) = \Phi_i$ for $i = 1, 2$.) Now since $\hat{\gamma}_1^{-1} g_1$ is in the closure of $\mathbf{H}(\mathbb{Q})$ in $\mathbf{H}(\mathbb{A}_f)$, there is a $\zeta \in \mathbf{H}(\mathbb{Q})$ such that $\hat{\gamma}_1^{-1} g_1\zeta \in \Omega$ leading to $g_1\zeta \in \hat{\gamma}_1\, \Omega = \hat{\Phi}\, \Omega \subset \Omega$. Since $g_1\zeta \in \mathbf{G}_1(\mathbb{Q}), \gamma_1 = g_1\zeta \in \Omega \cap \mathbf{G}_1(\mathbb{Q}) = \Phi_1$. Analogously $\gamma_2 = \zeta^{-1}g_2 \in \Phi_2$ so that $\gamma = \gamma_1\gamma_2$. Hence the lemma.

2.15. We will now apply these considerations to a special situation. Let k be a totally real number field and ∞ its set of archimedean valuations ($k_v \simeq \mathbb{R}$ for all $v \in \infty, k_v$ denoting the completion at v). We assume that $|\infty| \geq 2$. Let f be a quadratic form on a vector space E over k of dimension $n + 1$, where $n \geq 6$. We assume that E admits a basis $\mathcal{B} = (e_0, e_1, \ldots, e_n)$ such that the following conditions hold:

(i) For $x_i \in k, 0 \leq i \leq n, f(\sum_{0\leq i\leq n} x_i e_i) = \sum_{0\leq i\leq n} u_i x_i^2$, where $u_i \in k, 0 \leq i \leq n$.

(ii) For $i > 0, u_i$ is positive in every $k_v, v \in \infty$.

(iii) u_0 is positive in k_v for all $v \in \infty \backslash v_0$, for some v_0 and u_0 is negative in k_{v_0}.

Let $\mathbf{G}'$ denote the $k-$algebraic group $SO(f)$, the special orthogonal group of the quadratic form f. Let $\mathcal{B}_i, i = 1, 2$ be subsets of $\mathcal{B}$ containing e_0 and of cardinality $n_i + 1$. Assume further that the cardinality of $\mathcal{B}_1 \cap \mathcal{B}_2$ is $m + 1 = n_1 + n_2 - n + 1$. Let f_i denote the restriction of f to the k span of $\mathcal{B}_i$, and $\mathbf{G}_i^{'}$ the special orthogonal group $SO(f_i)$ of the quadratic form f_i. We then have natural inclusions $\mathbf{G}_i' \hookrightarrow \mathbf{G}'$ of $k-$algebraic groups. We also set $\mathbf{H}' = \mathbf{G}_1' \cap \mathbf{G}_2'$; then $\mathbf{H}'$ is precisely the special orthogonal group of the restriction g of f to the $k-$linear span F of $\mathcal{B}_1 \cap \mathcal{B}_2 = \mathcal{C}$. Let $\mathbf{G} = R_{k/\mathbb{Q}}\mathbf{G}', \mathbf{G}_i = R_{k/\mathbb{Q}}\mathbf{G}_i'$ and $\mathbf{H} = R_{k/\mathbb{Q}}\mathbf{H}'$. We will now fix an ideal $\underline{a}' \neq 0$ and groups $\wedge$ and Φ as in 2.9 for the groups $\mathbf{G}, \mathbf{G}_1, \mathbf{G}_2$ above. With this choice of $\mathbf{G}, \mathbf{G}_1, \mathbf{G}_2, \underline{a}'$ and Φ we have

2.16. Lemma. *The triple* $(\mathbf{G}_1, \mathbf{G}_2, \mathbf{H})$ *as above satisfies the condition in Lemma 2.12.*

Proof. $H^1(\mathbb{Q}, \mathbf{H})$ (resp. $H^1(\mathbb{Q}, \mathbf{G}_i), i = 1, 2$) is naturally isomorphic to $H^1(k, \mathbf{H}')$ (resp. $H^1(k, \mathbf{G}_i'), i = 1, 2$). The Galois cohomology set $H^1(k, \mathbf{H}')$ (resp. $H^1(k, \mathbf{G}_i'), i = 1, 2$) can be interpreted as the set of isomorphism classes of non-degenerate quadratic forms in $m + 1$ (resp $n_i + 1, i = 1, 2$) variables with the same discriminant as g (resp $f_i, i = 1, 2$). The natural map $H^1(k, \mathbf{H}') \to H^1(k, \mathbf{G}_i')$ in the context of this interpretation is the map which associates to each quadratic q form in $(m + 1)$ variables the form $q \perp \alpha_i$ (=orthogonal direct sum of q and α_i) where α_i is the form in $n_i - m$ variables given by the restriction of f to the $k-$span of $\mathcal{B}_i \backslash \mathcal{B}_{i+1}((i+1)$ taken (mod 2)$), i = 1, 2$. Now according to a well known " Cancellation Theorem" due to Witt, if q, q' are quadratic forms in $(m + 1)$ variables such that $q + \alpha_i \simeq q' + \alpha_i$, then $q \simeq q'$. It is not difficult to conclude from this that the map $H^1(k, \mathbf{H}') \to H^1(k, \mathbf{G}_i')$ is injective. Lemma 2.16 is now immediate.

2.17. Let $\tilde{\mathbf{G}}' \xrightarrow{\pi} \mathbf{G}'$ be the (two sheeted) spin covering of $\mathbf{G}'$. Let $\tilde{\mathbf{G}}'_i = \pi^{-1}(\mathbf{G}_i')$ and $\tilde{\mathbf{H}}' = \pi^{-1}(\mathbf{H}')$. One then has the following commutative diagram with exact rows of k-algebraic groups:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_2 & \longrightarrow & \tilde{\mathbf{H}}' & \xrightarrow{\pi} & \mathbf{H}' & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mu_2 & \longrightarrow & \tilde{\mathbf{G}}'_i & \xrightarrow{\pi} & \mathbf{G}'_i & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mu_2 & \longrightarrow & \tilde{\mathbf{G}}' & \xrightarrow{\pi} & \mathbf{G}' & \longrightarrow & 1
\end{array}$$

The kernel of π is isomorphic to the multiplicative group of order 2 over k which is denoted μ_2. This leads to the corresponding Galois cohomology exact sequences embedded in a commutative diagram:

$$\begin{array}{ccccl}
\tilde{\mathbf{H}}'(k) & \longrightarrow & \mathbf{H}'(k) & \xrightarrow{\delta} & H^1(k,\mu_2) \simeq (k^*)/(k^*)^2 \\
\downarrow & & \downarrow & & \| \\
\tilde{\mathbf{G}}'_i(k) & \longrightarrow & \mathbf{G}'_i(k) & \xrightarrow{\delta} & H^1(k,\mu_2) \\
\downarrow & & \downarrow & & \| \\
\tilde{\mathbf{G}}'(k) & \longrightarrow & \mathbf{G}'(k) & \xrightarrow{\delta} & H^1(k,\mu_2)
\end{array}$$

Let $k^+ = \{x \in k^* \mid x > 0 \text{ in every } k_v, v \in \infty\backslash\{v_0\}\}$. We assert that if $m \geq 3$, then $\delta(\mathbf{H}'(k)) = \delta(\mathbf{G}'_i(k)) = \delta(\mathbf{G}'(k)) = k^+/(k^*)^2$. This is seen as follows. $\delta(\mathbf{H}'(k))$ (resp. $\delta(\mathbf{G}'_i(k))$, resp. $\delta(\mathbf{G}'(k))$) is the subgroup $k^*/(k^*)^2$ generated by non-zero values of the quadratic form g (resp. f_i, resp. f) on the $k-$vector space [17]. By the Hasse principle [17] g (resp. f_i, resp. f) takes a value x in k if and only if it takes the value over k_v for all valuations v of k. Now since $m \geq 3$ if v is non-archimedean, g(resp. f_i, resp. f) takes every non-zero value over k_v[17]. When $v = v_0$ again g(resp. f_i, resp. f) takes every value in $k^*_{v_0}$ since g (resp. f_i, resp. f) is isotropic over k_{v_0}. For $v \in \infty\backslash\{v_0\}$, g (resp. f_i, resp. f) takes every positive value in k_v. Thus we have

2.18. Lemma. $\delta(\mathbf{H}'(k)) = \delta(\mathbf{G}'_i(k)) = \delta(\mathbf{G}'(k)) = k^+/(k^*)^2$.

The next result is a well known theorem due to Kneser (see [23]: Chapter 7).

2.19. Lemma. *Let* $\tilde{\mathbf{G}}$ *(resp.* $\tilde{\mathbf{G}}_i$*, resp.* $\tilde{\mathbf{H}}$*) be the* $\mathbb{Q}-$*algebraic group* $R_{k/\mathbb{Q}}\tilde{\mathbf{G}}'$ *(resp.* $R_{k/\mathbb{Q}}\tilde{\mathbf{G}}'_i$*, resp.* $R_{k/\mathbb{Q}}\tilde{\mathbf{H}}'$*). Then* $\tilde{\mathbf{G}}(\mathbb{Q})$ *(resp.* $\tilde{\mathbf{G}}_i(\mathbb{Q})$*, resp.* $\tilde{\mathbf{H}}(\mathbb{Q})$*) is dense in* $\tilde{\mathbf{G}}(\mathbb{A}_f)$ *(resp.* $\tilde{\mathbf{G}}_i(\mathbb{A}_f)$*, resp.* $\tilde{\mathbf{H}}(\mathbb{A}_f)$*) provided that* $n \geq 2$ *(resp.* $n_i \geq 2$*, resp.* $m \geq 2$*).*

2.20. Lemma. *Let $\mathbf{G}, \mathbf{G}_1, \mathbf{G}_2, \mathbf{H}$ be as in 2.15. Then there is an ideal $\underline{a} \neq 0$ in $\mathbb{Z}$ such that $\Gamma(\underline{a})$ (resp. $\Gamma_i(\underline{a}), i = 1, 2$, resp. $\triangle(\underline{a}) = \Gamma(\underline{a}) \cap \mathbf{H}(\mathbb{Q})$) is contained in the image of $\tilde{\mathbf{G}}(\mathbb{Q})$ (resp. $\tilde{\mathbf{G}}_i(\mathbb{Q}), i = 1, 2$, resp. $\tilde{\mathbf{H}}(\mathbb{Q})$).*

Proof. Denote by $\mathbf{B}$ any one of the groups, $\mathbf{G}, \mathbf{G}_i, \mathbf{H}$ and by $\tilde{\mathbf{B}}$ the spin cover of $\mathbf{B}$. We then have the exact sequence of $\mathbb{Q}-$groups

$$1 \to R_{k/\mathbb{Q}}\mu_2 \to \tilde{\mathbf{B}} \to \mathbf{B} \to 1$$

leading to the cohomology exact sequence

$$\tilde{\mathbf{B}}(\mathbb{Q}) \to \mathbf{B}(\mathbb{Q}) \xrightarrow{\delta} H^1(\mathbb{Q}, R_{k/\mathbb{Q}}\mu_2)$$

and $H^1(\mathbb{Q}, R_{k/\mathbb{Q}}\mu_2) \simeq H^1(k, \mu_2) \simeq k^*/(k^*)^2$. Now let $\wedge$ be one of the groups Γ, Γ_i, $\triangle = \Gamma \cap \mathbf{H}(\mathbb{Q})$ according as $\mathbf{B}$ is $\mathbf{G}$, $\mathbf{G}_i, \mathbf{H}$. Then $\wedge$ is finitely generated so that $\delta(\wedge)$ is a finite group. Now it is known [2] that the map $H^1(k, \mu_2) \to \prod_{v \in \mathcal{V}} H^1(k_v, \mu_2), \mathcal{V} =$ a complete set of inequivalent non-archimedean valuations, is injective. It follows that we can find a finite set $S' \subset \mathcal{V}$ such that

$$\delta(\Gamma) \to \prod_{v \in S'} H^1(k_v, \mu_2)$$

is injective. Let S be the set of valuations of $\mathbb{Q}$ lying below S'. For $w \in S$, let $\mathbf{B}^+(\mathbb{Q}_w)$ be the image $\tilde{\mathbf{B}}(\mathbb{Q}_w)$ in $\mathbf{B}(\mathbb{Q}_w)$. Then $\mathbf{B}^+(\mathbb{Q}_w)$ is an open subgroup of $\mathbf{B}(\mathbb{Q}_w)$. Let $\underline{a} \neq 0$ be an ideal in $\mathbb{Z}$ so chosen that $\wedge(\underline{a}) \subset \mathbf{B}^+(\mathbb{Q}_w)$ for all $w \in S$: since $\mathbf{B}^+(\mathbb{Q}_w)$ is open $\mathbf{B}(\mathbb{Q}_w)$ for $w \in S$, such an ideal $\underline{a}$ exists. It is now clear that $\underline{a}$ is a nonzero ideal with the desired properties.

2.21. Let $\mathbf{G}, \mathbf{G}_1, \mathbf{G}_2, \mathbf{H}$ be as above. Let $\mathcal{D} = \mathcal{B} \backslash \mathcal{C} \cup \{e_0\}$. Let $\mathbf{L}'$ be the special orthogonal group of the quadratic form $h = f$ restricted to the span of $\mathcal{D}$. Let $\mathbf{L} = R_{k/\mathbb{Q}}\mathbf{L}'$. Now $\mathbf{H}' \cap \mathbf{L}'$ is trivial so that the natural map $\mathbf{H}' \times \mathbf{L}' \to \mathbf{H}'\mathbf{L}' (\subset \mathbf{G})$ is an isomorphism of $k-$ varieties. It follows that every element u of $\mathbf{H}'\mathbf{L}'$ is uniquely expressible as a product $h\ell$ with $h \in \mathbf{H}'$ and $\ell \in \mathbf{L}'$, and if $u = h\ell \in \mathbf{G}'(k)$ then $h \in \mathbf{H}'(k), \ell \in \mathbf{L}'(k)$. Thus one has $\mathbf{H}(\mathbb{Q})\mathbf{L}(\mathbb{Q}) = (\mathbf{HL})(\mathbb{Q}) = \mathbf{HL} \cap \mathbf{G}(\mathbb{Q})$. Now choose an ideal $\underline{a}'$ as in 2.9 taking $\mathbf{G}_1 = \mathbf{H}$ and $\mathbf{G}_2 = \mathbf{L}$ in Lemma 2.10. In view of Lemma 2.20, one can assume that $\Gamma(\underline{a}') \subset \mathbf{G}(\mathbb{Q})^+ =$ Image $\tilde{\mathbf{G}}(\mathbb{Q})$ in $\mathbf{G}(\mathbb{Q})$. Let $\Gamma(\underline{a}') = \wedge(\mathbf{H}, \mathbf{L})$ and let $\wedge(\mathbf{H})$ (resp. $\wedge(\mathbf{L})$) be the group $\wedge(\mathbf{H}, \mathbf{L}) \cap \mathbf{H}(\mathbb{Q})$ (resp. $\wedge(\mathbf{H}, \mathbf{L}) \cap \mathbf{L}(\mathbb{Q})$). We assume moreover that $\underline{a}'$ has been so chosen that it satisfies the conditions of 2.9 also for $\mathbf{G}_1, \mathbf{G}_2$ and further that if $\mathbf{B}$ is one of the groups $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}, \mathbf{L}, \mathbf{H}$ and $\tilde{\mathbf{B}}$ its spin cover, then $\Gamma(\underline{a}') \cap \mathbf{B} \subset \mathbf{B}(\mathbb{Q})^+ =$ Image $\tilde{\mathbf{B}}(\mathbb{Q})$ in $\mathbf{B}(\tilde{\mathbf{B}}$ is the spin cover of $\mathbf{B}$). Now from Lemma 2.10, we know that there is an ideal $\underline{b} \neq 0, \underline{b} = \underline{b}(\wedge(H, L))$ such that for any subgroup $\Psi \subset \wedge(\mathbf{H})\Gamma(\underline{b})$ containing $\wedge(\mathbf{H})\Gamma(\underline{c})$ for some non-zero ideal $\underline{c}$, we have

$$\Psi \cap \mathbf{HL} = \widehat{\Psi}(\mathbf{H})\widehat{\Psi}(\mathbf{L}),$$

where $\widehat{\Psi}(\mathbf{H})$ (resp. $\widehat{\Psi}(\mathbf{L})$) is the closure of $\Psi(\mathbf{H}) = \Psi \cap \mathbf{H}(\mathbb{Q})$ (resp. $\Psi(\mathbf{L}) = \Psi \cap \mathbf{L}(\mathbb{Q})$) in $\mathbf{G}(\mathbb{A}_f)$. On the other hand, we have

$$\Psi \cap \mathbf{HL} \subset \mathbf{H}(\mathbb{Q})\mathbf{L}(\mathbb{Q}).$$

Thus if $\gamma \in \Psi \cap \mathbf{HL}$,

$$\gamma = \widehat{\gamma}(\mathbf{H})\widehat{\gamma}(\mathbf{L}) = g(\mathbf{H})g(\mathbf{L})$$

with $\widehat{\gamma}(\mathbf{H})$ (resp. $\widehat{\gamma}(\mathbf{L})$, resp. $g(\mathbf{H})$, resp. $g(\mathbf{L})$) in $\widehat{\Psi}(\mathbf{H})$ (resp. $\widehat{\Psi}(\mathbf{L})$, resp. $\mathbf{H}(\mathbb{Q})$, resp. $\mathbf{L}(\mathbb{Q})$). Hence

$$g(\mathbf{H})^{-1}\widehat{\gamma}(\mathbf{H}) = g(\mathbf{L})\widehat{\gamma}(\mathbf{L})^{-1} \in (\mathbf{H} \cap \mathbf{L})(\mathbb{A}_f) = \{1\},$$

so that $g(\mathbf{H}) = \widehat{\gamma}(\mathbf{H}) \in \widehat{\Psi}(\mathbf{H}) \cap \mathbf{H}(\mathbb{Q}) = \Psi(\mathbf{H})$ and similarly $g(\mathbf{L}) \in \Psi(\mathbf{L})$. We conclude from this that $\gamma = \gamma(\mathbf{H})\gamma(\mathbf{L})$ with $\gamma(\mathbf{H}) \in \Psi(\mathbf{H})$ and $\gamma(\mathbf{L}) \in \Psi(\mathbf{L})$. We record this as

2.22. Lemma. *Fix an ideal $\underline{a}' \neq 0$ in $\mathbb{Z}$ such that the following conditions hold: let $\mathbf{B}$ denote any one of the groups $\mathbf{G}, \mathbf{G}_i, i = 1, 2, \mathbf{H}$ or $\mathbf{L}$; then if $\Phi \subset \Gamma(\underline{a}')$ is any subgroup of finite index, one has:*

(i) Φ *is torsion free.*

(ii) $\Phi \cap \mathbf{B}(\mathbb{Q}) \subset \mathbf{B}(\mathbb{Q})^+ =$ *Image* $\tilde{\mathbf{B}}(\mathbb{Q})$ *in* $\mathbf{B}(\mathbb{Q})$.

(iii) *Let $\mathbf{M}'$ be the subgroup of $\mathbf{G}' = SO(f)$ which stabilises the 1-dimensional subspace of E spanned by e_0 and $\mathbf{M} = R_{k/\mathbb{Q}}\mathbf{M}'$. Let $K = \mathbf{M}(\mathbb{R})$. Then*

$$\Phi \cap \mathbf{H}(\mathbb{R})K\mathbf{L}(\mathbb{R}) \subset \mathbf{HL}$$

and

$$\Phi \cap \mathbf{G}_1(\mathbb{R})K\mathbf{G}_2(\mathbb{R}) \subset \mathbf{G}_1\mathbf{G}_2.$$

Then there is an ideal $\underline{b'} \neq 0$ *contained in* $\underline{a'}$ *such that for any subgroup* Ψ *of* $\Gamma(\underline{a'})$ *satisfying*

$$\Gamma(\underline{c})(\Gamma(\underline{a'}) \cap \mathbf{H}(\mathbb{Q})) \subset \Psi \subset \Gamma(\underline{b'})(\Gamma(\underline{a'}) \cap \mathbf{H}(\mathbb{Q})),$$

for some nonzero ideal $\underline{c}$*, also satisfies*

$$\Psi \cap \mathbf{H}(\mathbb{R}) K \mathbf{L}(\mathbb{R}) \subset (\Psi \cap \mathbf{H}(\mathbb{Q}))(\Psi \cap \mathbf{L}(\mathbb{Q})).$$

2.23. Remarks.

(i) We have shown that an ideal $\underline{a'}$ satisfying (i) - (iii) in the Lemma exists.

(ii) The group $\tilde{\mathbf{B}}(\mathbb{R})$ is connected. Consequently elements of $\mathbf{B}(\mathbb{Q})^+$ and hence $\Gamma(\underline{a'}) \cap \mathbf{B}(\mathbb{Q})^+$ act as orientation preserving automorphisms of the orientable manifold $K \cap \mathbf{B}(\mathbb{R}) \backslash \mathbf{B}(\mathbb{R})$.

(iii) $K \cap \mathbf{B}(\mathbb{R})$ is a maximal compact subgroup of $\mathbf{B}(\mathbb{R})$ and the natural map of $K \cap \mathbf{B}(\mathbb{R}) \backslash \mathbf{B}(\mathbb{R})$ is an imbedding of this symmetric space (of constant curvature) as a totally geodesic submanifold of $K \backslash G$ (which is itself a Riemannian symmetric space of constant curvature).

2.24. We now fix ideals $\underline{a'}$ and $\underline{b'}$ as in Lemma 2.22. Let $\wedge$ be a subgroup of $\Gamma(\underline{a'})$ with

$$\Gamma(\underline{b'})(\Gamma(\underline{a'}) \cap \mathbf{H}(\mathbb{Q})) \supset \wedge \supset \Gamma(\underline{c})(\Gamma(\underline{a'}) \cap \mathbf{H}(\mathbb{Q}))$$

for some non-zero ideal $\underline{c}$. Choose now an ideal $\underline{b} \neq 0$ contained in $\underline{a'}$ such that for any subgroup Φ of $\Gamma(\underline{a'})$ with

$$\Gamma(\underline{c})\wedge_1 \subset \Phi \subset \Gamma(\underline{b})\wedge_1,$$

where $\wedge_1 = \wedge \cap \mathbf{G}_1(\mathbb{Q}), \Phi \cap \mathbf{G}_1\mathbf{G}_2 \subset \widehat{\Phi}_1\widehat{\Phi}_2$ (recall that $\Phi_i = \Phi \cap \mathbf{G}_i(\mathbb{Q})$ and $\widehat{\Phi}_i =$ closure of Φ_i in $\mathbf{G}(\mathbb{A}_f)$). Now the triple $\mathbf{G}, \mathbf{G}_1, \mathbf{G}_2$ satisfy the conditions in Lemma 2.12 (see Lemma 2.16). Thus if $\gamma \in \mathbf{G}_1\mathbf{G}_2 \cap \Phi$, then $\gamma = g_1 g_2 = \widehat{\gamma}_1\widehat{\gamma}_2$ with $g_i \in \mathbf{G}_i(\mathbb{Q}), \widehat{\gamma}_i \in \widehat{\Phi}_i$. We assert that we can choose g_1 such that $\widehat{\gamma}_1^{-1} g_1$ is in the closure of $\mathbf{H}(\mathbb{Q})$. For this it suffices to show that $\widehat{\gamma}_1^{-1} g_1$ is in the image of $\tilde{\mathbf{H}}(\mathbb{A}_f)$ in $\mathbf{H}(\mathbb{A}_f)$. (See Lemma 2.19). Since $\widehat{\gamma}_1 \in$ Image $\tilde{\mathbf{G}}_1(\mathbb{A}_f) = \mathbf{G}_1(\mathbb{A}_f)^+$ as ensured by condition (ii) in Lemma 2.22, for every $p-$adic component $\widehat{\gamma}_1(p), p \in \mathcal{P}, \delta\widehat{\gamma}_1(p) = 0$ in

$H^1(\mathbb{Q}_p, \mu_2)$. If $\delta(\widehat{\gamma}_1(p)^{-1}g_1) = \delta(\widehat{\gamma}_1(p)^{-1})\delta(g_1) = 0$, then every $p-$adic component of $\widehat{\gamma}_1^{-1}g_1$ will be in the image of $\tilde{\mathbf{H}}(\mathbb{Q}_p)$ and hence $\widehat{\gamma}_1^{-1}g_1$ will be in the image of $\tilde{\mathbf{H}}(\mathbb{A}_f)$. Thus it suffices to show that $\gamma = u_1.u_2$ with $\delta(u_1) = 0$, since $\delta(\gamma) = 0$, this would mean $\delta(u_2) = 0$ as well. According to Lemma 2.18, there is an element ζ in $\mathbf{H}(\mathbb{Q})$ such that $\delta(g_1) = \delta(\zeta)$ and we need only set $u_1 = g_1\zeta^{-1}, u_2 = \zeta g_2$. We have thus shown

2.25. Theorem. *There exists an arithmetic subgroup $\Phi \subset \mathbf{G}(\mathbb{Q})$ such that the following conditions hold.*

(i) *Φ is torsion free.*

(ii) *$\Phi \cap \mathbf{B}(\mathbb{Q}) \subset \mathbf{B}(\mathbb{Q})^+ =$ Image $\tilde{\mathbf{B}}(\mathbb{Q})$ where $\mathbf{B}$ is one of the groups $\mathbf{G}, \mathbf{G}_1, \mathbf{G}_2, \mathbf{H}$ or $\mathbf{L}$ and $\tilde{\mathbf{B}}$ is the spin covering of $\mathbf{B}$.*

(iii) *Let $\mathbf{M}'$ be the subgroup of $\mathbf{G}' = SO(f)$ leaving the 1-dimensional subspace spanned by e_0 stable and $\mathbf{M} = R_{k/\mathbb{Q}}\mathbf{M}'$. Let $K = \mathbf{M}(\mathbb{R})$.*

Then $K \cap \mathbf{B}(\mathbb{R})$ is a maximal compact subgroup of $\mathbf{B}(\mathbb{R})$ for any $\mathbf{B}$ as above and we have

$$\Phi \cap \mathbf{G}_1(\mathbb{R})K\mathbf{G}_2(\mathbb{R}) \subset (\Phi \cap \mathbf{G}_1(\mathbb{Q}))(\Phi \cap \mathbf{G}_2(\mathbb{Q}))$$

and

$$\Phi \cap \mathbf{H}(\mathbb{R})K\mathbf{L}(\mathbb{R}) \subset (\Phi \cap \mathbf{H}(\mathbb{Q}))(\Phi \cap \mathbf{L}(\mathbb{Q})).$$

2.26. Corollary. *Let $X = K\backslash\mathbf{G}(\mathbb{R}), X_i = K \cap \mathbf{G}_i(\mathbb{R})\backslash\mathbf{G}_i(\mathbb{R}), Z = X_1 \cap X_2 = K \cap \mathbf{H}(\mathbb{R})\backslash\mathbf{H}(\mathbb{R})$ and $Y = K \cap \mathbf{L}(\mathbb{R})\backslash\mathbf{L}(\mathbb{R})$. If Φ is as in the theorem, then the natural maps $X_i/\Phi \cap \mathbf{G}_i(\mathbb{Q}) \to X/\Phi$, $Z/\Phi \cap \mathbf{H}(\mathbb{Q}) \to X/\Phi$ and $Y/\Phi \cap \mathbf{L}(\mathbb{Q}) \to X/\Phi$ are (totally geodesic isometric) imbeddings of compact orientable manifolds in the orientable manifold X/Φ. Moreover $X_1/\Phi\cap\mathbf{G}_i(\mathbb{Q})$ and $X_2/\Phi\cap\mathbf{G}_2(\mathbb{Q})$ intersect transversally in the connected submanifold $Z/\Phi \cap \mathbf{H}(\mathbb{Q})$ while $Z/\Phi \cap \mathbf{H}(\mathbb{Q})$ intersects $Y/\Phi \cap \mathbf{L}(\mathbb{Q})$ transversally in a single point viz. the identity double coset $K\Phi$ in $K\backslash\mathbf{G}(\mathbb{R})/\Phi$.*

Proof. Let $p \in (X_1/\Phi \cap \mathbf{G}_1(\mathbb{Q})) \cap (X_2/\Phi \cap \mathbf{G}_2(\mathbb{Q}))$. Then there are elements $g_1 \in \mathbf{G}_1(\mathbb{R}), g_2 \in \mathbf{G}_2(\mathbb{R}), k \in K$ and $\gamma \in \Phi$ with $kg_2\gamma^{-1} = g_1^{-1}$ with Kg_1^{-1} (as well as Kg_2) projecting to p. This means that $\gamma = g_1kg_2$ i.e., $\gamma \in \Phi\cap\mathbf{G}_1(\mathbb{R})K\mathbf{G}_2(\mathbb{R})$. By the theorem we conclude that $\gamma = \gamma_1\gamma_2$ with $\gamma_i \in \mathbf{G}_i(\mathbb{Q}) \cap \Phi$. Thus $kg_2\gamma_2^{-1} = g_1^{-1}\gamma_1$, which means that p is the image of $p' = Kg_2\gamma_2^{-1} = Kg_1^{-1}\gamma_1$. Clearly $p' \in X_1 \cap X_2 = Z$ so that $p \in$ Image Z in X/Φ. That the intersection of $X_1/\Phi \cap \mathbf{G}_1(\mathbb{Q})$ and $X_2/\Phi \cap$

$G_2(\mathbb{Q})$ is transversal follows from the transversality of the intersection of X_1 and X_2. Next suppose $q \in (Z/\Phi \cap \mathbf{H}(\mathbb{Q})) \cap (Y/\Phi \cap \mathbf{L}(\mathbb{Q}))$; then there exist $h \in \mathbf{H}(\mathbb{R}), k \in K$ and $\ell \in \mathbf{L}(\mathbb{R})$ such that $k\ell\theta^{-1} = h^{-1}$ with $\theta \in \Phi$, i.e., $\theta \in H(\mathbb{R})KL(\mathbb{R}) \cap \Phi$. By the theorem $\theta = \theta(\mathbf{H})\theta(\mathbf{L})$ with $\theta(\mathbf{H}) \in \mathbf{H}(\mathbb{Q}) \cap \Phi$ and $\theta(\mathbf{L}) \in \mathbf{L}(\mathbb{Q}) \cap \Phi$. Arguing exactly as above, we see that q is in the image of $Z \cap Y$ which is the identity coset in $X = K\backslash G(\mathbb{R})$. Thus $Z \cap Y$ is precisely the identity double coset. That the intersection is transversal follows from the fact that Y and Z intersect transversally. This completes the proof of the theorem.

References

[1] S. I. Al'ber, *Spaces of mappings into manifold of negative curvature*, Dokl. Akad. Nauk. SSSR **178** (1968) 13–16.

[2] E. Artin & J. Tate, *Classfield theory*, Benjamin, 1968.

[3] D. Birkes, *Orbits of linear algebraic groups*, Ann. of Math. **93** (1971) 459–475.

[4] A. Borel & Harish-Chandra, *Arthmetic subgroups of algebraic groups*, Ann. of Math. **75** (1962) 485–535.

[5] C. Chevalley, *Deux Théorimes d'arithmatiques*, J. Math. Soc. Japan **3** (1951) 36–44.

[6] K. Corlette, *Archimedean superrigidity and hyperbolic geometry*, Ann. of Math. **135** (1992) 165–182.

[7] J. Eells & L. Lemaire, Selected topics in harmonic maps, CBMS Regional Conf. Ser. 50, Amer. Math. Soc., Providence, RI, 1983.

[8] J. Eells & J.H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964) 109–160.

[9] F. T. Farrell & L. E. Jones, *Negatively curved manifolds with exotic smooth structures*, J. Amer. Math. Soc. **2** (1989) 899–908.

[10] ———, *Some non-homeomorphic harmonic homotopy equivalences*, Bull. London Math. Soc. **28** (1996) 177–180.

[11] F. T. Farrell, L. E. Jones & P. Ontaneda, *Hyperbolic manifolds with negatively curved exotic triangulations in dimensions larger than five*, J. Differential Geom. **48** (1998) 319–322.

[12] ———, *Examples of non-homeomorphic harmonic maps between negatively curved manifolds*, Bull. London Math. Soc. **30** (1998) 295–296.

[13] P. Hartman, *On homotopic harmonic maps*, Canad. J. Math. **19** (1967) 673–687.

[14] L. Hernández, *Kähler manifolds and 1/4-pinching*, Duke Math. J. **62** (1991) 601–611.

[15] J. Jost & S.-T. Yau, *Harmonic maps and superrigidity*, Proc. Sympos. Pure Math. **54** (Amer. Math. Soc., Providence, R.I., 1993), 245–280.

[16] R. C. Kirby & L. C. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Ann. of Math. Stud. No. **88** Princeton Univ. Press, Princeton, 1977.

[17] M. Kneser, *Lectures on Galois cohomology of classical groups*, Lecture Notes, Tata Institute, Mumbai.

[18] S. Lang, *Algebraic number theory*, Springer.

[19] J.J. Millson & M.S. Raghunathan, *Geometric construction of cohomology of arithmetic groups. I*, Proc. Indian Acad. Sci. **90** (1981) 103–123.

[20] N. Mok, Y.-T. Siu & S.-K. Yeung, *Geometric superrigidity*, Invent. Math. **113** (1993) 57–83.

[21] P. Ontaneda, *Hyperbolic manifolds with negatively curved exotic triangulations in dimension six*, J. Differential Geom. **40** (1994) 7–22.

[22] V. P. Platonov, *The problem of strong approximation and the Kneser-Tits conjecture for algebraic groups*, Math. USSR Izvestiya **3** (1969) 1139–1147.

[23] V. P. Platonov & Rapinchuk, *Algebraic Groups and Number Theory*, Academic Press, 1994.

[24] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer, 1972.

[25] ———, *The Congruence subgroup problem*, Proc. Hyderabad Conf. Algebraic Groups, Manoj Prakshan, (India), 465–494.

[26] Rohlfs & J. Schwermer, *Intersection number of special cycles*, J. Amer. Math. Soc. **6** (1993) 755–778.

[27] J. Sampson, *Some properties and applications of harmonic mappings*, Ann. Sci. École Norm. Sup. **11** (1978) 211–228.

[28] M. Scharlemann & L. Siebenmann, *The Hauptvermutung for smooth singular homeomorphisms*, Manifolds Tokyo 1973, (University of Tokyo Press, Tokyo, 1975), 85–91.

[29] R. Schoen & S.-T. Yau, *On univalent harmonic maps between surfaces*, Invent. Math. **44** (1978) 265–278.

[30] Y.-T. Siu, *The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds*, Ann. of Math. **112** (1980) 73–111.

[31] C.W. Stark, *Surgery theory and infinite fundamental groups*, Ann. of Math. Vol. 1, No. 145, 275–305, Princeton Univ. Press, Princeton, NJ, 2000.

[32] S.-T. Yau, *Seminar on differential geometry*, Ann. of Math. Stud., No. 102, Princeton Univ. Press, Princeton, NJ, 1982.

[33] S.-T. Yau & F. Zheng, *Negatively 1/4-pinched riemannian metric on a compact Kähler manifold*, Invent. Math. **103** (1991) 527–535.

SUNY, Binghamton
UFPE, Recife, Brazil
Tata Institute of Fundamental Research, India